

A NOTE ON THE POLYNOMIAL-LIKE ITERATIVE EQUATIONS ORDER

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ABSTRACT. We show that, under reasonable assumptions, two negative roots can be eliminated from the characteristic equation of a polynomial-like iterative equation. This result gives a new case where we may lower the order of such an equation.

1. INTRODUCTION

Let n be a positive integer and $I \subset \mathbb{R}$ be an interval. We are interested in so-called polynomial-like iterative equations, namely, equations of the form

$$(1) \quad a_n f^n(x) + \dots + a_1 f(x) + a_0 x = 0,$$

where f^k stands for the k -fold iterate of a self-mapping unknown function $f: I \rightarrow I$, the coefficients a_n, \dots, a_1, a_0 are given real numbers and $a_0 \neq 0$. In general, it is difficult to find all continuous functions satisfying equation (1) even in the case $n = 3$; a partial solution in this case was given in [7] and the complete solution for $n = 2$ was presented in [3, 5]. One of methods for finding solutions to equation (1) involves lowering its order. Such results were obtained in [1, 6, 9] (see Theorem 2 below); the present paper contains a new result in this spirit. For a similar investigation concerning non-homogenous polynomial-like iterative equations, where zero on the right-hand side is replaced by an arbitrary continuous function, see, e.g. [8].

We shall recall some basic properties of solutions to polynomial-like equations. Assume that a continuous function $f: I \rightarrow I$ satisfies equation (1). It can be easily shown that f is injective (see, e.g. [1, Lemma 2.1]) and therefore monotone. Assuming that $f(x) = rx$ we obtain the so-called characteristic equation of (1):

$$(2) \quad a_n r^n + \dots + a_1 r + a_0 = 0.$$

This equation may also be considered as the characteristic equation of the recurrence relation

$$(3) \quad a_n x_{j+n} + \dots + a_1 x_{j+1} + a_0 x_j = 0$$

in which the sequence $(x_j)_{j \in \mathbb{N}_0}$ is obtained in the following way: We choose $x_0 \in I$ arbitrarily and put $x_j = f(x_{j-1})$ for $j \in \mathbb{N}$. It is easy to see that f satisfies (1) if and only if $(x_j)_{j \in \mathbb{N}_0}$ satisfies (3).

2010 *Mathematics Subject Classification.* 39B12.

Key words and phrases. Continuous solution, Iterate, Polynomial-like iterative equation, Recurrence relation.

Since the function f is monotone, the sequence $(x_j)_{j \in \mathbb{N}_0}$ is either monotone (in the case of increasing f) or anti-monotone (in the case of decreasing f). By *anti-monotone* we mean that the expression $(-1)^j(x_{j+1} - x_j)$ does not change its sign when j runs through \mathbb{N}_0 . Consider $y_0 \in I$ and define a sequence $(y_j)_{j \in \mathbb{N}_0}$ in the same way as we did for x_0 . Similarly, the sequence $(x_j - y_j)_{j \in \mathbb{N}_0}$ has a constant sign, in the case of increasing f , or alternates in sign, in the case of decreasing f . Let us note that the sequence $(x_{j+1} - y_j)_{j \in \mathbb{N}_0}$ also has the same property.

In the case where f is surjective (and hence bijective) we can consider the dual equation

$$(4) \quad a_0 f^n(x) + \dots + a_{n-1} f(x) + a_n x = 0.$$

Putting $f^{-n}(x)$ in place of x we see that f satisfies (1) if and only if f^{-1} satisfies (4). We can also extend the above defined sequence $(x_j)_{j \in \mathbb{N}_0}$ to the whole \mathbb{Z} by setting $x_{-j} = f^{-1}(x_{-j+1})$ for $j \in \mathbb{N}$. Then relation (3) is satisfied for all $j \in \mathbb{Z}$.

For the theory of linear recurrence relations we refer the reader, for instance, to [2, §3.2]. We shall recall only the most significant theorem in this matter. In order to do this and simplify the writing we introduce the following notation: For a given polynomial $c_n r^n + \dots + c_1 r + c_0$ we denote by $\mathcal{R}(c_n, \dots, c_0)$ the collection $\{(r_1, k_1), \dots, (r_p, k_p)\}$ of all pairs of pairwise distinct (complex) roots r_1, \dots, r_p and their multiplicities k_1, \dots, k_p , respectively. Here and throughout the present paper by a polynomial we mean a polynomial with real coefficients. Note that in the introduced notation $k_1 + \dots + k_p$ equals the degree of $c_n r^n + \dots + c_1 r + c_0$ and by writing $(\mu, k), (\bar{\mu}, k) \in \mathcal{R}(c_n, \dots, c_0)$ we mean μ to be non-real.

Theorem 1. *Assume that*

$$\mathcal{R}(a_n, \dots, a_0) = \{(\lambda_1, l_1), \dots, (\lambda_p, l_p), (\mu_1, m_1), (\bar{\mu}_1, m_1), \dots, (\mu_q, m_q), (\bar{\mu}_q, m_q)\}.$$

Then a real-valued sequence $(x_j)_{j \in \mathbb{N}_0}$ is a solution to (3) if and only if it is given by

$$x_j = \sum_{k=1}^p A_k(j) \lambda_k^j + \sum_{k=1}^q (B_k(j) \cos j \phi_k + C_k(j) \sin j \phi_k) |\mu_k|^j \quad \text{for } j \in \mathbb{N}_0,$$

where A_k is a polynomial whose degree equals at most $l_k - 1$ for $k = 1, \dots, p$ and B_k, C_k are polynomials whose degrees equal at most $m_k - 1$, with ϕ_k being an argument of μ_k , for $k = 1, \dots, q$.

It is worth mentioning that the above theorem is also valid for sequences defined on the whole \mathbb{Z} . We shall use this fact in the proof of our main result.

2. THE MAIN RESULT

It was observed by Matkowski and Zhang in [4] that if a polynomial $b_m r^m + \dots + b_1 r + b_0$ divides $a_n r^n + \dots + a_1 r + a_0$ and f satisfies

$$(5) \quad b_m f^m(x) + \dots + b_1 f(x) + b_0 x = 0,$$

then f satisfies also (1). One of methods for solving equation (1) involves a partial converse of this statement. More precisely, we want to find a divisor of the polynomial $a_n r^n + \dots + a_1 r + a_0$ such that the corresponding polynomial-like iterative equation of lower order is

satisfied. Known results, concerning elimination of non-real roots or real roots of opposite sign, are listed below.

Theorem 2. (i) [1, Thm. 3.3] (cf. [6, Thm. 5] and [9, Thm. 1]) *Assume that*

$$\mathcal{R}(a_n, \dots, a_0) = \{(\lambda_1, l_1) \dots, (\lambda_p, l_p), (\mu_1, k_1), (\overline{\mu}_1, k_1), \dots, (\mu_q, k_q), (\overline{\mu}_q, k_q)\}.$$

If $|\lambda_1| \leq \dots \leq |\lambda_p| < |\mu_1| \leq \dots \leq |\mu_q|$, then a continuous function $f: I \rightarrow I$ satisfies equation (1) if and only if it satisfies (5) with

$$\mathcal{R}(b_m, \dots, b_0) = \{(\lambda_1, l_1) \dots, (\lambda_p, l_p)\}.$$

(ii) [1, Thms. 4.1 and 4.2] *Assume that*

$$\mathcal{R}(a_n, \dots, a_0) = \{(r_1, k_1), (r_2, k_2), (\lambda_1, l_1), \dots, (\lambda_p, l_p)\}.$$

Let also $|r_1| < |\lambda_1| \leq \dots \leq |\lambda_p| < |r_2|$ and r_1, r_2 be real with $r_1 r_2 < 0$; say $r_i > 0$ and $r_j < 0$. Then a continuous increasing surjection $f: I \rightarrow I$ satisfies equation (1) if and only if it satisfies (5) with

$$\mathcal{R}(b_m, \dots, b_0) = \{(r_i, k_i), (\lambda_1, l_1), \dots, (\lambda_p, l_p)\}.$$

If $r_i \neq 1$, then a continuous decreasing surjection $f: I \rightarrow I$ satisfies equation (1) if and only if it satisfies (5) with

$$\mathcal{R}(b_m, \dots, b_0) = \{(r_j, k_j), (\lambda_1, l_1), \dots, (\lambda_p, l_p)\}.$$

If $r_i = 1$, then a continuous decreasing surjection $f: I \rightarrow I$ satisfies equation (1) if and only if it satisfies (5) with

$$\mathcal{R}(b_m, \dots, b_0) = \{(1, 1), (r_j, k_j), (\lambda_1, l_1), \dots, (\lambda_p, l_p)\}.$$

Those results were proved by examining the asymptotic behaviour of the sequence of consecutive iterates of the unknown function at a given point. Using a similar approach we obtain our new result, concerning elimination of negative roots, which reads as follows.

Theorem 3. *Assume that*

$$\mathcal{R}(a_n, \dots, a_0) = \{(r_1, k_1), (r_2, k_2), (\lambda_1, l_1), \dots, (\lambda_p, l_p)\}.$$

If $|r_2| < |\lambda_1| \leq \dots \leq |\lambda_p| < |r_1|$ and r_1, r_2 are real with $r_1 < -1 < r_2 < 0$, then a continuous surjection $f: I \rightarrow I$ satisfies equation (1) if and only if it satisfies equation (5) with

$$\mathcal{R}(b_m, \dots, b_0) = \{(r_i, k_i), (\lambda_1, l_1), \dots, (\lambda_p, l_p)\},$$

where $i = 1$ or $i = 2$.

Proof. Choose $x \in I$ arbitrarily. Define a sequence $(x_j)_{j \in \mathbb{Z}}$ by putting $x_0 = x$, $x_j = f(x_{j-1})$ and $x_{-j} = f^{-1}(x_{-j+1})$ for $j \in \mathbb{N}$. Then relation (3) is satisfied for all $j \in \mathbb{Z}$. Therefore, by Theorem 1, we have

$$x_j = A(j)r_1^j + F(j) + B(j)r_2^j \quad \text{for } j \in \mathbb{Z},$$

where A, B are polynomials and F stands for the part of solution to (3) for which the roots $\lambda_1, \dots, \lambda_p$ are responsible. We shall show that either $A \equiv 0$ or $B \equiv 0$.

For an indirect proof suppose that both polynomials A and B are non-zero. Denote by s and t degrees of A and B , respectively. Similarly, let a and b be the leading coefficients of A and B .

Since

$$\begin{aligned} (-1)^j(x_{j+1} - x_j) &= (A(j+1)r_1 - A(j))|r_1|^j \\ &\quad + (-1)^j(F(j+1) - F(j)) + (B(j+1)r_2 - B(j))|r_2|^j, \end{aligned}$$

we have

$$(6) \quad \lim_{j \rightarrow -\infty} \frac{(-1)^j(x_{j+1} - x_j)}{|j|^t \cdot |r_2|^j} = (-1)^t(r_2 - 1)b,$$

$$(7) \quad \lim_{j \rightarrow \infty} \frac{(-1)^j(x_{j+1} - x_j)}{j^s \cdot |r_1|^j} = (r_1 - 1)a.$$

This shows that the sequence $(x_j)_{j \in \mathbb{Z}}$ cannot be monotone (in fact, this shows that it cannot be monotone when either $A \not\equiv 0$ or $B \not\equiv 0$); consequently, f cannot be increasing. Thus f is decreasing.

According to the above observation the expression $(-1)^j(x_{j+1} - x_j)$ has a constant sign when j runs through \mathbb{Z} . Combining this fact with equations (6) and (7), we conclude that a and $(-1)^t b$ have the same sign. Further, since f^2 is increasing, the expression

$$\begin{aligned} x_{2j+2} - x_{2j} &= (A(2j+2)r_1^2 - A(2j))|r_1|^{2j} \\ &\quad + F(2j+2) - F(2j) + (B(2j+2)r_2^2 - B(2j))|r_2|^{2j} \end{aligned}$$

also has a constant sign. Similarly, we have

$$\lim_{j \rightarrow -\infty} \frac{x_{2j+2} - x_{2j}}{|2j|^t \cdot |r_2|^{2j}} = (-1)^t(r_2^2 - 1)b, \quad \lim_{j \rightarrow \infty} \frac{x_{2j+2} - x_{2j}}{(2j)^s \cdot |r_1|^{2j}} = (r_1^2 - 1)a.$$

As a result, a and $(-1)^t b$ are of opposite sign; a contradiction. Therefore, $A \equiv 0$ or $B \equiv 0$. Using Theorem 1 once again we conclude that the assertion holds for a fixed $x \in I$. It remains to show that elimination of the root r_1 or r_2 does not depend on x .

Consider $y \in I$ and define a sequence $(y_j)_{j \in \mathbb{Z}}$ in the same way as we did for x . Suppose, for the sake of a contradiction, that $x_j = A(j)r_1^j + F(j)$ and $y_j = G(j) + B(j)r_2^j$ for $j \in \mathbb{Z}$ with non-zero polynomials A and B (F and G stand for the terms for which the roots $\lambda_1, \dots, \lambda_p$ are responsible). As before, let s, t be the degrees and a, b be the leading coefficients of A and B , respectively. Since f monotonically decreases, the sequence $(x_j - y_j)_{j \in \mathbb{Z}}$ alternates in sign. Thus the expression

$$(-1)^j(x_j - y_j) = A(j)|r_1|^j + (-1)^j(F(j) - G(j)) - B(j)|r_2|^j$$

has a constant sign. Further, we have

$$\lim_{j \rightarrow -\infty} \frac{(-1)^j(x_j - y_j)}{|j|^t \cdot |r_2|^j} = (-1)^{t+1}b, \quad \lim_{j \rightarrow \infty} \frac{(-1)^j(x_j - y_j)}{j^s \cdot |r_1|^j} = a$$

which means that a and $(-1)^{t+1}b$ have the same sign. Repeating this reasoning with the sequence $(x_{j+1} - y_j)_{j \in \mathbb{Z}}$ we conclude that a and $(-1)^{t+1}b$ have opposite signs. The obtained contradiction ends the proof. \square

Remark 4. Since the equation $2f^2(x) + 5f(x) + 2x = 0$ is satisfied by $f(x) = -2x$ and $f(x) = -\frac{1}{2}x$, in general, it cannot be decided which root from r_1 and r_2 may be eliminated. Therefore, Theorem 3 states that equation (1) is actually equivalent to an alternative of two equations of lower order.

Remark 5. It is worth mentioning that if $I = \mathbb{R}$, then f is necessarily bijective (see [9, Lemma 1]). Therefore, the assumption of surjectivity in Theorems 2 and 3 is satisfied automatically in this case.

Remark 6. Using quoted results and Theorem 3 the order of equation (1) can be essentially lowered in many important cases. However, some cases still remains open. For instance, it is unknown whether non-real roots may be eliminated from characteristic equation (2) without any additional assumptions (cf. [1, Section 6] and [9, Section 6]).

Acknowledgement. The research was supported by the University of Silesia Mathematics Department (Iterative Functional Equations and Real Analysis program).

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